J. Fluid Mech. (1970), *vol.* 41, *part* 4, *pp.* 689–705
Printed in Great Britain

The flow fields in and around a droplet moving axially within a tube

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(Received 13 August 1969 and in revised form 19 November 1969)

A solution is presented for the flow field in and around a single spherical droplet or bubble moving axially at an arbitrary radial location, within a long circular tube. In the tube there is viscous fluid flowing with a constant Poiseuillian velocity distribution far from the droplet.

The settling velocity of the droplet or bubble is

$$U = \frac{2(\rho_i-\rho_e)ga^2}{9\mu_e}\frac{1+\alpha}{\frac{2}{3}+\alpha}\left[1-\frac{2+3\alpha}{3(1+\alpha)}\left(\frac{a}{R_0}\right)f\left(\frac{b}{R_0}\right)\right]+U_0\left[1-\left(\frac{b}{R_0}\right)^2 - \frac{2\alpha}{2+3\alpha}\left(\frac{a}{R_0}\right)^2\right]+O\left(\frac{a}{R_0}\right)^3.$$

This is a general equation and it contains as special cases the familiar solutions of Stokes, Hadamard–Rybczynski, Brenner & Happel, Greenstein & Happel and Haberman & Sayre.

The function describing the deviation of the interface from sphericity is solved and an iterative procedure is suggested for obtaining higher order solutions.

1. Introduction and existing solutions

1.1. *Introduction*

A considerable volume of literature has developed in the field of the dynamics of multiphase flow, including the specific problem of viscous flow around a solid particle or droplet.

In the present work, some of the difficulties involved are discussed and a solution presented for the flow field in and around a single spherical droplet or bubble moving in a pipe where fluid flows with constant Poiseuillian velocity.

Previous theoretical studies have been based upon either the macroscopic or homogeneous approach or upon the microscopic approach. The former is based on a phenomenological description of a homogeneous fluid, whose physical properties are determined by the properties of its components. The macroscopic approach yields relatively easy solutions, which may differ from the single-phase solution only by the difference in physical properties. It is, however, difficult to ascertain whether these properties have been correctly defined and whether the solution is physically meaningful. Furthermore, the flow field is not completely solved, since the boundary conditions can be defined only on the walls of the conduit and do not include the boundaries of the discontinuous phase. Due to these limitations, solutions based on the homogeneous model may be of practical use, but do not provide an insight into the underlying physical phenomena.

In contrast, the microscopic analysis solves the actual flow field by considering the full set of boundary conditions. A complete mathematical description of multiphase systems is not yet available, mainly because of the interaction between such factors as particle and residence time distributions, turbulent intensities, presence of surface active impurities and other phenomena. As a consequence it is usual to deal with a single particle or to employ the free-surface cell model.

The free-surface cell model describes a representative particle enclosed in an imaginary cell, on which some boundary conditions are defined. It is further assumed *a priori* that the particles are arranged in a certain fixed array in the flow field. The main difficulties with this model are to choose correctly the geometry of the cell and to define appropriate boundary conditions on its surface.

The theories dealing with a single particle suspended in a flow field do not consider the interaction between particles. However, these theories may yield the minimum distance between particles which is necessary for ignoring the interaction between them. If the distance between particles is larger than this minimum, one can assume that the interaction between them is negligible and the solution of the flow field for a single particle is applicable to the whole flow field.

1.2. *Existing solutions*

The existing solutions for a single particle or droplet suspended in a creeping flow field will only be highlighted here. Most solutions are well summarized by Happel & Brenner (1965).

For all existing solutions the following are assumed: the fluids are homogeneous isothermal, Newtonian and of constant density; the flow is laminar and in steady state; the flow around the droplet or particle is creeping, i.e. $Re \ll 1$ such that the inertia terms in the equations may be neglected. The observer is assumed to be in an inertial co-ordinate system. Surface active agents are assumed to be absent in most solutions.

With these suppositions the equations of motion reduce to the Stokes equations for creeping motion.

The boundary conditions are usually defined as follows:

(i) no-slip on any solid surface,

(ii) matching velocities and stresses across any two-phase interface,

(iii) a prescribed velocity far from the bubble or particle.

A tacit assumption is frequently made that the droplet is spherical. This assumption may lead to difficulties and is further discussed in § 2.2.

The Stokes equations, subject to these boundary conditions, have been solved for various cases. Stokes solved the settling velocity of a solid particle moving in an unbound viscous fluid. Hadamard and Rybczynski independently solved the velocity fields inside and outside of a droplet moving in an unbound viscous fluid. Levich (1962) extended their solution for the case when surface-active agents are present. Brenner & Happel (1958) solved the approximate flow field around a solid particle moving at an arbitrary location in Poiseuillian flow in a tube, using the method of 'reflexion'.

Haberman & Sayre (1958) solved the equations, in a precise manner, for a solid particle moving along the centreline of a tube. They also solved approximately the flow field around a droplet, under similar conditions. Goldsmith & Mason (1962) discussed the lateral migration, perpendicular to the streamlines of the Poiseuille flow, of solid particles. They reported that liquid drops migrated towards the tube axis, at very low Reynolds numbers, and investigated this phenomenon in terms of the deformation of the drop and the non-uniformity of the velocity gradient. Chaffey, Brenner & Mason (1965) presented a first-order theory for the migration of a neutrally buoyant drop suspended in a Couette flow. Recently, Greenstein & Happel (1968) evaluated the wall effects on the motion of a particle suspended in Poiseuillian flow in a tube.

2. Statement of the problem and boundary conditions

2.1. *Statement of the problem*

The problem considered herein is that of a single droplet or bubble moving in the z direction with constant velocity U parallel to the longitudinal axis of an infinitely long circular cylinder. In the cylinder there is a viscous fluid flowing in the z direction, with constant Poiseuillian velocity distribution and with a maximum velocity U_0. The centre of the droplet is situated at a distance b from the cylinder axis.

The co-ordinate systems employed here are cylindrical (R, Φ, Z) and spherical (r, θ, ϕ). The origin of the spherical co-ordinate system coincides with the centre of the droplet. The co-ordinate systems are depicted in figure 1.

The fluid around the droplet flows in creeping motion, i.e. with small Reynolds number (where $Re = 2a|U_0(1-b^2/R_0^2)-U|/\nu$, ν is the kinematic viscosity). The inertial terms in the equations of motion may therefore be neglected. Thus, the equations of motion to be satisfied are

$$\nabla^2\mathbf{v} = \frac{1}{\mu_e}\nabla p_e, \tag{1}$$

and

$$\nabla^2\mathbf{u} = \frac{1}{\mu_i}\nabla p_i, \tag{2}$$

where $\mathbf{v}$ is the velocity vector of the continuous medium and $\mathbf{u}$ is the velocity vector inside the droplet, in terms of a co-ordinate system which moves with the droplet. The pressure p includes the potential gravity field. The subscripts 'e' and 'i' refer to the properties exterior to the droplet and interior to it, respectively.

The equations of continuity are

$$\nabla.\mathbf{v} = 0, \tag{3}$$

and

$$\nabla.\mathbf{u} = 0. \tag{4}$$

2.2. The boundary conditions

The boundary conditions to be employed (neglecting surface-active agents) are as follows: far from the droplet the flow field is Poiseuillian. In terms of the co-ordinate system which moves with the droplet, this becomes,

$$\text{at} \quad z = \pm\infty, \quad \mathbf{v} = \mathbf{v}_\infty = \{U_0[1-(R/R_0)^2]-U\}\mathbf{k}, \tag{5a}$$

where $\mathbf{k}$ is the unit vector in the z direction.

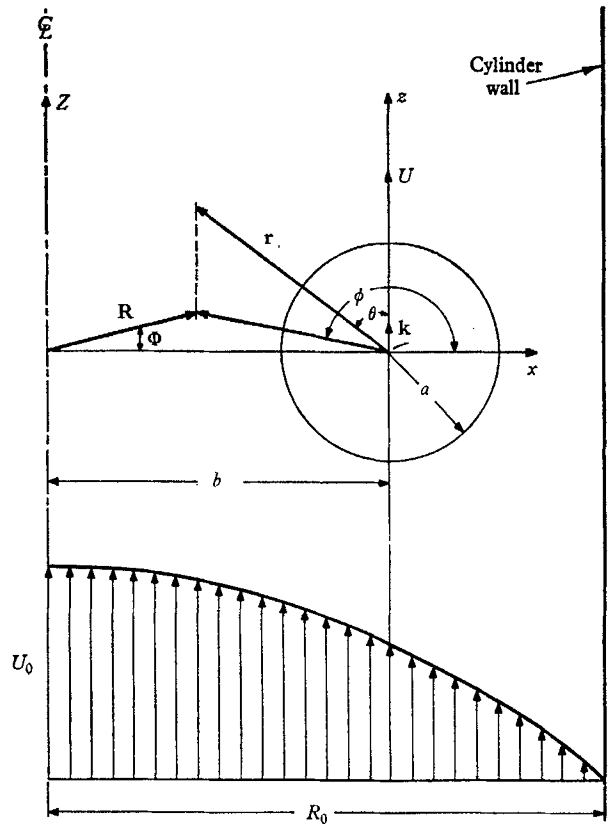

FIGURE 1. The geometry and co-ordinate systems used in the analysis.

In the spherical co-ordinate system this velocity is expressed as

$$\mathbf{v}_\infty = \left\{U_0\left[1-\left(\frac{r}{R_0}\right)^2\sin^2\theta-\left(\frac{b}{R_0}\right)^2-\frac{2rb}{R_0^2}\sin\theta\cos\phi\right]-U\right\}\mathbf{k}. \tag{5b}$$

The usual condition of non-slip at the wall of the cylinder in terms of the same co-ordinate system is,

$$\text{at} \quad R = R_0, \quad \mathbf{v} = -U\mathbf{k}. \tag{6}$$

At the interface of the droplet and the continuous medium the following is assumed:

(i) continuity of the tangential velocity vectors, i.e.

$$\mathbf{v}_{(t)} = \mathbf{u}_{(t)}, \tag{7}$$

where $\mathbf{v}_{(t)}$ and $\mathbf{u}_{(t)}$ are two-dimensional vectors;

(ii) vanishing of the normal component of the velocity vector, i.e.

$$v_n = u_n = 0; \tag{8}$$

(iii) continuity of the tangential components of the normal stress

$$\boldsymbol{\tau}_{(n)(t)} = \boldsymbol{\pi}_{(n)(t)}, \tag{9}$$

where $\boldsymbol{\tau}_{(n)(t)}$ is the tangential component of the normal stress vector $\boldsymbol{\tau}_{(n)}$ inside the droplet;

(iv) the relation between the outside and inside values of the normal components of the normal stress vectors is

$$\tau_{(n)(n)} = \pi_{(n)(n)} - \sigma\left(\frac{1}{R_1} + \frac{1}{R_2}\right), \tag{10}$$

where σ is the surface tension and $R_1(\theta, \phi)$ and $R_2(\theta, \phi)$ are the radii of curvature of the droplet.

Thus, a set of seven boundary conditions on the interface is obtained:

(7), 2 boundary conditions,
(8), 2 boundary conditions,
(9), 2 boundary conditions,
(10), 1 boundary condition.

In addition, there is a boundary condition at infinity, namely (5), and the requirement that the solution be finite everywhere.

The solution of the problem subject to these boundary conditions should yield the velocity field around the droplet and the general equation describing the geometry of the interface. This equation makes this solution differ from the one for a solid particle.

The first six boundary conditions, together with the known velocity distribution far from the droplet, suffice to determine explicitly the six unknown velocity components for a known geometry of the interface. The pressure field inside and outside of the droplet can be calculated from (1) and (2) up to a constant. The constant involved in the outside pressure field is determined from the known pressure far from the droplet. The constant involved in the interior pressure field is determined from the seventh boundary condition on the interface. This boundary condition yields a constant value if, and only if, the geometry of the interface is used correctly. If a wrong geometry is used, the last boundary condition yields a functional relationship indicating an inconsistency in the boundary conditions.

Hadamard and Rybczynski already encountered this difficulty. However, rather than using the seventh boundary condition for determining the constant in the expression for the interior pressure field, they used it for calculating the settling velocity U. This value should be obtained from a simple force balance on the droplet. Their solution is, however, numerically correct since for a uniform flow field the droplet can be shown to be indeed spherical. Haberman & Sayre (1958) encountered the same difficulty, and, since their flow field was Poiseuillian, the droplet was not spherical and the difficulty was not resolved.

3. The solution

The mathematical treatment for solving simultaneously the flow fields and the equation of the interface is excessively difficult. Therefore, an iterative procedure is adopted here. First, the droplet is postulated to be spherical with radius 'a', and the flow fields inside the droplet and outside it are solved, using only six of the boundary conditions. Later the function describing the deviation of the droplet from sphericity is determined using the seventh boundary condition. The newly determined interface may then be used for calculating the flow fields of the second iteration. (This iterative procedure was not completed in the present work.) Therefore, the solution presented herein should be considered as a first approximation of a much more complex problem.

The solution of the first iteration is based on the method of 'reflexion' described by Happel & Brenner (1965, p. 299). Thus, the solution consists of the sum of a series of velocity fields, all of which satisfy equations (1) and (3) for the velocity field of the continuous medium and equations (2) and (4) for the velocity field interior to the droplet. Each of the solutions partially satisfies the boundary conditions as follows:

Zeroth reflexion:

$$\mathbf{v}^{(0)} = \mathbf{v}_\infty = \left\{U_0\left[1-\left(\frac{R}{R_0}\right)^2\right]-U\right\}\mathbf{k}, \tag{11}$$

$$p_e^{(0)} = p_\infty = \left(-\frac{4\mu_e U_0}{R_0^2}+\rho_e g\right)z+\text{const.} \tag{12}$$

First reflexion:

at the interface,
$$\mathbf{u}^{(1)} = \mathbf{v}^{(1)}+\mathbf{v}^{(0)}, \tag{13a}$$

$$u_r^{(1)} = v_r^{(1)}+v_r^{(0)} = 0, \tag{13b}$$

$$\boldsymbol{\tau}_{(r)}^{(1)} = \boldsymbol{\pi}_{(r)}^{(1)} - \sigma\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\mathbf{t}_r; \tag{13c}$$

at $z = \pm\infty$,
$$\mathbf{v}^{(1)} = 0; \tag{13d}$$

where $\boldsymbol{\pi}_{(r)}^{(1)}$ is based on the velocity $\mathbf{v}^{(1)}+\mathbf{v}^{(0)}$ and acts on an area whose direction is $\mathbf{t}_r$, and where subscript r indicates a normal to the interface (this subscript replaces 'n' of the previous boundary conditions, since the solutions for a spherical droplet are now sought).

Second reflexion:

$$\mathbf{v}^{(2)} = \left\{\begin{matrix}-\mathbf{v}^{(1)} & \text{at} & R = R_0, \\ 0 & \text{at} & z=\pm\infty.\end{matrix}\right\} \tag{14}$$

Third reflexion:
$$\mathbf{u}^{(3)} = \mathbf{v}^{(3)}+\mathbf{v}^{(2)}, \tag{15a}$$

$$u_r^{(3)} = v_r^{(3)}+v_r^{(2)} = 0, \tag{15b}$$

$$\boldsymbol{\tau}_{(r)}^{(3)} = \boldsymbol{\pi}_{(r)}^{(3)} - \sigma\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\mathbf{t}_r; \tag{15c}$$

at $z = \pm\infty$,
$$\mathbf{v}^{(3)} = 0; \tag{15d}$$

where $\boldsymbol{\pi}_{(r)}^{(3)}$ is based on the velocity $\mathbf{v}^{(3)}+\mathbf{v}^{(2)}$, etc., etc.

The fields $\mathbf{v}$ and $\mathbf{u}$ satisfying the boundary conditions (5) through (9) are then obtained by superposition of the solutions, i.e.

$$\mathbf{v} = \mathbf{v}^{(0)} + \mathbf{v}^{(1)} + \mathbf{v}^{(2)} + \mathbf{v}^{(3)} + \dots, \tag{16}$$

$$\mathbf{u} = \mathbf{u}^{(1)} + \mathbf{u}^{(3)} + \dots, \tag{17}$$

with similar summations of the pressures p_e and p_i.

3.1. *The first reflected field*

The general solution of the first reflected field $\mathbf{v}^{(1)}$, using Lamb's (1945, p. 595) solution in spherical co-ordinates is

$$\mathbf{v}^{(1)} = \sum_{n=1}^{\infty} \left\{ \nabla \times (\mathbf{r}\chi_{-n-1}) + \nabla\Phi_{-n-1} - \frac{(n-2)}{2n(2n-1)\mu_e} r^2 \nabla p_{-n-1} + \frac{(n+1)}{n(2n-1)\mu_e} \mathbf{r} p_{-n-1} \right\}, \tag{18}$$

and

$$p_e^{(1)} = \sum_{n=0}^{\infty} p_{-n-1}, \tag{19}$$

where only the negative harmonics are used in order to obtain $\mathbf{v}^{(1)} \rightarrow 0$ at $z = \pm\infty$, to satisfy boundary condition (5).

For the flow field interior to the droplet, we get similarly

$$\mathbf{u}^{(1)} = \sum_{n=1}^{\infty} \left\{ \nabla \times (\mathbf{r}\chi_n) + \nabla\Phi_n + \frac{(n+3)}{2(n+1)(2n+3)\mu_i} r^2 \nabla p_n - \frac{n}{(n+1)(2n+3)\mu_i} \right\} \mathbf{r} p_n, \tag{20}$$

and

$$p_i^{(1)} = \sum_{n=0}^{\infty} p_n, \tag{21}$$

where only the positive solid harmonics are used in order to obtain finiteness of the velocity field at $r = 0$.

Here $\mathbf{r}$ is the radius vector drawn from the sphere origin and χ_{-n-1}, Φ_{-n-1}, p_{-n-1}, χ_n, Φ_n and p_n are solid spherical harmonics of degree $-n-1$ and n, respectively. These solid spherical harmonics are now solved using a transformed set of boundary conditions presented in appendix A.

Substituting (18) and (20) into the transformed boundary conditions (A 10) to (A 15), and using the orthogonality properties of the Legendre polynomials, we obtain six sets of equations with six sets of unknowns. The six sets of unknowns are the solid spherical harmonics χ_n, Φ_n, p_n, χ_{-n-1}, Φ_{-n-1}, p_{-n-1} $(n = 1, 2, 3, \dots)$.

In order to proceed with the solution we define some spherical harmonics in the following way:

$$\left.\begin{aligned}
p_{-1} &= \mu_e A_{-1} r^{-1} P_0(\cos\theta), & \Phi_{-2} &= a^3 B_{-2} r^{-2} P_1(\cos\theta),\\
p_{-2} &= \mu_e a A_{-2} r^{-2} P_1(\cos\theta), & \Phi_{-3} &= a^4 B_{-3} r^{-3} \cos\phi P_2^1(\cos\theta),\\
p_{-3} &= \mu_e a^2 A_{-3} r^{-3} \cos\phi P_2^1(\cos\theta), & \Phi_{-4} &= a^5 B_{-4} r^{-4} P_3(\cos\theta),\\
p_{-4} &= \mu_e a^3 A_{-4} r^{-4} P_3(\cos\theta), & \Phi_1 &= B_1 r P_1(\cos\theta),\\
p_1 &= \mu_i a^{-2} A_1 r P_1(\cos\theta), & \Phi_2 &= a^{-1} B_2 r^2 \cos\phi P_2^1(\cos\theta),\\
p_2 &= \mu_i a^{-3} A_2 r^2 \cos\phi P_2^1(\cos\theta), & \Phi_3 &= a^{-2} B_3 r^3 P_3(\cos\theta),\\
p_3 &= \mu_i a^{-4} A_3 r^3 P_3(\cos\theta), & \chi_1 &= a^{-1} C_1 r \sin\phi P_1^1(\cos\theta),\\
p_0 &= \mu_i a^{-1} A_0 P_0(\cos\theta), & \chi_{-2} &= a^2 C_{-2} r^{-2} \sin\phi P_1^1(\cos\theta).
\end{aligned}\right\} \tag{22}$$

All other spherical harmonics do not contribute to the expressions for the velocities.

Substituting (22) in the six sets of equations, and equating the coefficients of the Legendre polynomials of the same order (i.e. by setting $n = 1, 2, 3$), we obtain five independent groups of equations (all other equations yield a set of homogeneous independent equations). These equations may be used for determining the coefficients of the Legendre polynomials in (22) as follows:†

$$\left.\begin{aligned}
A_{-2} &= \frac{2+3\alpha}{2+2\alpha}\left[U-U_0\left(1-\frac{b^2}{R_0^2}-\frac{2\alpha}{2+3\alpha}\frac{a^2}{R_0^2}\right)\right],\\
A_{-3} &= \tfrac{2}{3}U_0\frac{ab}{R_0^2}\frac{2+5\alpha}{1+\alpha},\\
A_{-4} &= -\tfrac{1}{2}U_0\left(\frac{a}{R_0}\right)^2\frac{2+7\alpha}{1+\alpha},\\
A_1 &= \frac{5}{1+\alpha}\left[U_0\left(1-\frac{b^2}{R_0^2}-2\frac{a^2}{R_0^2}\right)-U\right],\\
A_2 &= -7U_0\frac{ab}{R_0^2}\frac{1}{1+\alpha},\\
A_3 &= 6U_0\left(\frac{a}{R_0}\right)^2\frac{1}{1+\alpha};\\
B_{-2} &= \frac{\alpha}{4(1+\alpha)}\left[U-U_0\left(1-\frac{b^2}{R_0^2}-\frac{6\alpha-4}{5\alpha}\frac{a^2}{R_0^2}\right)\right],\\
B_{-3} &= \tfrac{1}{3}U_0\frac{ab}{R_0^2}\frac{\alpha}{1+\alpha},\\
B_{-4} &= -\tfrac{1}{4}U_0\left(\frac{a}{R_0}\right)^2\frac{\alpha}{1+\alpha},\\
B_1 &= -\frac{1}{2(1+\alpha)}\left[U_0\left(1-\frac{b^2}{R_0^2}-2\frac{a^2}{R_0^2}\right)-U\right],\\
B_2 &= \tfrac{1}{2}U_0\frac{ab}{R_0^2}\frac{1}{1+\alpha},\\
B_3 &= -\tfrac{1}{3}U_0\left(\frac{a}{R_0}\right)^2\frac{1}{1+\alpha},\\
C_1 &= U_0\frac{ab}{R_0^2},\\
C_{-2} &= 0;
\end{aligned}\right\} \qquad (23)$$

where $\alpha = \mu_i/\mu_e$.

Thus, the expressions for the velocity fields for the first reflexion are

$$\mathbf{v}^{(1)} = \nabla(\Phi_{-2}+\Phi_{-3}+\Phi_{-4})+\frac{r^2}{\mu_e}\nabla(\tfrac{1}{2}p_{-2}-\tfrac{1}{30}p_{-4})+\frac{\mathbf{r}}{\mu_e}(2p_{-2}+\tfrac{1}{2}p_{-3}+\tfrac{4}{15}p_{-4}), \qquad (24)$$

$$\mathbf{u}^{(1)} = \nabla\times(\mathbf{r}\chi_1)+\nabla(\Phi_1+\Phi_2+\Phi_3)+\frac{r^2}{\mu_i}\nabla(\tfrac{1}{5}p_1+\tfrac{5}{42}p_2+\tfrac{1}{12}p_3) -\frac{\mathbf{r}}{\mu_i}(\tfrac{1}{10}p_1+\tfrac{2}{21}p_2+\tfrac{1}{12}p_3); \qquad (25)$$

† For footnote see facing page.

and the corresponding pressure fields are

$$p_e^{(1)} = p_{-1}+p_{-2}+p_{-3}+p_{-4}, \tag{26}$$

$$p_i^{(1)} = p_0+p_1+p_2+p_3, \tag{27}$$

where the solid spherical harmonics are defined by (22) and (23).

The drag on the droplet due to the velocity field of the first reflexion can be found by using the generalization of Faxen's law‡ as given by Hetsroni & Haber (1969), viz.

$$\mathbf{F}^{(1)} = 2\pi\mu_e a\frac{2+3\alpha}{1+\alpha}[\mathbf{v}^{(0)}]_0+\pi\mu_e a^3\frac{\alpha}{1+\alpha}[\nabla^2\mathbf{v}^{(0)}]_0, \tag{28}$$

where the subscripted parentheses indicate that the functions are evaluated at the centre of the droplet. Substitution of (5) into (28) yields

$$\mathbf{F}^{(1)} = 2\pi\mu_e a\frac{2+3\alpha}{1+\alpha}\left[U_0\left(1-\frac{b^2}{R_0^2}-\frac{2\alpha}{2+3\alpha}\frac{a^2}{R_0^2}\right)-U\right]\mathbf{k}, \tag{29}$$

which is the drag force on a spherical droplet suspended in an unbounded Poiseuillian flow field.

3.2. *The second reflected field*

Henceforth we limit ourselves to an approximation of the velocity field in (24), namely

$$\mathbf{v}^{(1)} \simeq \frac{1}{2\mu_e}\nabla(r^2p_{-2})+\frac{\mathbf{r}}{\mu_e}p_{-2}. \tag{30}$$

The solution of the second reflected field $\mathbf{v}^{(2)}$ is identical to the one given by Happel & Brenner (1965, p. 305), except for their constant H, which should be replaced by the coefficient A_{-2} of this work. Notice that the force due to the second reflected field (and all other even-numbered reflexions) is zero.

3.3. *The third reflexion*

To evaluate the drag force acting on the droplet due to the third reflexion $\mathbf{F}^{(3)}$, we use equation (28), using $\mathbf{v}^{(2)}$ as the unperturbed velocity field, i.e.

$$\mathbf{F}^{(3)} = 2\pi\mu_e a\frac{2+3\alpha}{1+\alpha}[\mathbf{v}^{(2)}]_0+\pi\mu_e a^3\frac{\alpha}{1+\alpha}[\nabla^2\mathbf{v}^{(2)}]_0.$$

† Note that the solution of the coefficients could readily be obtained by using the general solution of Hetsroni & Haber (1969). In this case these coefficients are

$$\alpha_1^0 = -\tfrac{2}{5}U_0\left(\frac{a}{R_0}\right)^2, \quad \beta_3^0 = \tfrac{2}{5}U_0\left(\frac{a}{R_0}\right)^2,$$

$$\beta_1^0 = U_0\left[1-\left(\frac{b}{R_0}\right)^2\right]-U, \quad \hat{\gamma}_1^1 = 2U_0\frac{ab}{R_0^2},$$

$$\beta_2^1 = -\tfrac{2}{3}U_0\frac{ab}{R_0^2},$$

where the α's, β's and γ are coefficients depending only on the unperturbed velocity field and on the geometry.

‡ For the first reflexion only the drag can be evaluated also by using $F = 4\pi\nabla(r^3p_{-2})$ (Happel & Brenner 1965, p. 67).

Using the expression for $\mathbf{v}^{(2)}$ from Happel & Brenner one obtains

$$\mathbf{F}^{(3)} = 2\pi\mu_e a \left\{\frac{1}{3}\left(\frac{2+3\alpha}{1+\alpha}\right)^2 \left[U_0\left(1-\frac{b^2}{R_0^2}\right)-U\right]\frac{a}{R_0} f\left(\frac{b}{R_0}\right) + O\left(\frac{a}{R_0}\right)^3\right\}\mathbf{k}, \quad (31)$$

where $f(b/R_0)$ was evaluated by Famularo (1962, table 1), and in Happel & Brenner (1965, p. 309). More recent values of this function are given by Greenstein & Happel (1968). Notice that (31) is correct to $O(a/R_0)^3$, even though the approximation (30) has been used for $\mathbf{v}^{(1)}$. This was proved by Haber (1969) and previously by Greenstein (1966).

The external velocity can be calculated from equation (31), using the method of Happel & Brenner (p. 312). This velocity is

$$\mathbf{v}^{(3)} = \frac{1}{2\mu_e}\nabla(r^2 p_{-2}^{(3)}) + \frac{\mathbf{r}}{\mu_e} p_{-2}^{(3)}, \quad (32)$$

where

$$p_{-2}^{(3)} = \mu_e a A_{-2}^{(3)} r^{-2}\cos\theta, \quad (33)$$

and where

$$A_{-2}^{(3)} = \frac{(2+3\alpha)^2}{6(1+\alpha)^2}\left[U - U_0\left(1-\frac{b^2}{R_0^2}\right)\right]\frac{a}{R_0} f\left(\frac{b}{R_0}\right). \quad (34)$$

3.4. *The fourth and fifth reflexions*

The velocity field of the fourth reflexion $\mathbf{v}^{(4)}$ is obtained in a similar way to that used for $\mathbf{v}^{(2)}$, with $A_{-2}^{(3)}$ replacing A_{-2}.

The drag force due to the fifth reflexion can be obtained based on $\mathbf{v}^{(4)}$, as before

$$\mathbf{F}^{(5)} = 2\pi\mu_e a \left\{\frac{1}{9}\left(\frac{2+3\alpha}{1+\alpha}\right)^3 \left[U_0\left(1-\frac{b^2}{R_0^2}\right)-U\right]\left(\frac{a}{R_0}\right)^2 f^2\left(\frac{b}{R_0}\right) + O\left(\frac{a}{R_0}\right)^3\right\}\mathbf{k}. \quad (35)$$

3.5. *The terminal settling velocity and the drag*

Upon summing the individual drag forces of the first six reflexions one readily obtains

$$\mathbf{F} = \sum_{i=0}^{6} F^{(i)} = 2\pi\mu_e a \frac{2+3\alpha}{1+\alpha}\left\{\left[U_0\left(1-\frac{b^2}{R_0^2}\right)-U\right]\left[1+\left(\frac{a}{R_0}\right)\frac{2+3\alpha}{3+3\alpha} f\left(\frac{b}{R_0}\right)\right.\right.$$
$$\left.\left. + \left(\frac{a}{R_0}\right)^2\left(\frac{2+3\alpha}{3+3\alpha}\right)^2 f^2\left(\frac{b}{R_0}\right)\right] - \frac{2\alpha}{2+3\alpha} U_0\left(\frac{a}{R_0}\right)^2 + O\left(\frac{a}{R_0}\right)^3\right\}\mathbf{k}. \quad (36)$$

This is the drag force acting on a spherical droplet or bubble suspended in Poiseuillian flow in a tube, at a distance b from the centreline.

The terminal settling velocity can now be obtained from a force balance on the droplet

$$\mathbf{F} + \frac{4\pi}{3} a^3 g(\rho_i - \rho_e)\,\mathbf{k} = 0. \quad (37)$$

Substitution of (36) in (37) and using the approximation

$$(1+x+x^2) \simeq (1-x)^{-1} + O(x^3)$$

one obtains

$$U = \frac{2(\rho_i-\rho_e) g a^2}{3\mu_e}\frac{1+\alpha}{2+3\alpha}\left[1 - \frac{2+3\alpha}{3+3\alpha}\left(\frac{a}{R_0}\right) f\left(\frac{b}{R_0}\right)\right]$$
$$+ U_0\left[1-\left(\frac{b}{R_0}\right)^2 - \frac{2\alpha}{2+3\alpha}\left(\frac{a}{R_0}\right)^2\right] + O\left(\frac{a}{R_0}\right)^3. \quad (38)$$

This is a general equation for the terminal settling velocity of a spherical droplet or bubble moving axially at an arbitrary radial location within a long circular tube. In the tube there is viscous fluid flowing with a constant Poiseuillian velocity distribution. This general solution contains as special cases the familiar solution of Stokes ($U_0 = 0$, $\alpha = \infty$, $R_0 = \infty$) Hadamard–Rybczynski ($U_0 = 0$, $R_0 = \infty$), Brenner & Happel ($\rho_i = \rho_e$, $\alpha = \infty$), Greenstein & Happel ($\alpha = \infty$) and Haberman & Sayre ($\rho_i = \rho_e$, $b = 0$).

4. The equation of the interface

The solution of the flow fields, drag force and terminal settling velocity was completed for a spherical droplet. These were evaluated without making use of the seventh boundary condition, i.e. (10).

The first iteration is now completed by evaluating the deviation of the interface from sphericity. (This new interface may then be used for calculating the velocity fields of the second iteration.) This is done by using the seventh boundary condition.

The radius of a nearly spherical droplet may be represented by

$$r = a[1+\xi(\theta,\phi)], \tag{39}$$

where $|\xi(\theta,\phi)| \ll 1$. Following the procedure suggested by Hetsroni & Haber (1969), $\xi(\theta,\phi)$ is now described by a sum of surface harmonics as follows:

$$\xi(\theta,\phi) = \sum_{n=2}^{\infty}\sum_{m=0}^{n} [L_n^m \cos(m\phi) + \hat{L}_n^m \sin(m\phi)]\, P_n^m(\cos\theta), \tag{40}$$

where L_n^m and $\hat{L}_n^m$ were computed, in a general form, in terms of α_n^m, β_n^m, $\hat{\alpha}_n^m$ and $\hat{\beta}_n^m$. The α's and β's depend only on the unperturbed velocity field and on the geometry. The general solution of L_n^m was given as

$$L_n^m = \frac{1}{(n+1)\,n(n^2+n-2)(1+\alpha)}\left\{\frac{\alpha_n^m \mu_e}{\sigma}[(4n^3+6n^2+2n+3)\alpha + (4n^3+6n^2-4n-6)] + \frac{\beta_n^m \mu_e}{\sigma}[(4n^3+6n^2+2n-3)\,\alpha + (4n^3+6n^2-4n)]\right\}, \tag{41}$$

where $n \geqslant 2$. The coefficients $\hat{L}_n^m$ are given by an identical expression, with $\hat{\alpha}_n^m$ and $\hat{\beta}_n^m$ replacing α_n^m and β_n^m, respectively.

In order to determine the deviation function $\xi(\theta,\phi)$ to $O(a/R_0)^2$, one must examine also the wall effects on the deformation. That is, one must calculate the coefficients $\alpha_n^{m(0)}$, $\alpha_n^{m(2)}$, $\alpha_n^{m(4)}$, ..., $\beta_n^{m(0)}$, $\beta_n^{m(2)}$, $\beta_n^{m(4)}$, etc., to the desired accuracy (the parenthesized superscripted number indicates the reflected velocity field used for determining the coefficients).

The zeroth reflexion, i.e. $\mathbf{v}^{(0)} = \mathbf{v}_\infty$, yields the following coefficients:

$$\beta_2^{1(0)} = -\tfrac{2}{3}U_0\frac{ab}{R_0^2}; \quad \beta_3^{0(0)} = \tfrac{2}{5}U_0\left(\frac{a}{R_0}\right)^2. \tag{42}$$

It can be shown (Haber 1969) that the second reflected field $\mathbf{v}^{(2)}$ yields only one coefficient of the desired order, namely $\beta_2^{1(2)}$. All other coefficients are of

$O(a/R_0)^3$ and higher. Furthermore, the fourth reflected field and higher reflexions have no contribution towards L_n^m of $O(a/R_0)^2$. Therefore, it is necessary to calculate only $\beta_2^{1(2)}$.

In appendix C it is shown that

$$a^{-1}\beta_2^{1(2)} S_2(\theta, \phi) = \mathbf{t}_r \mathbf{t}_r : [\nabla \mathbf{v}^{(2)}]_{r=0}. \tag{43}$$

Using the expression for $\mathbf{v}^{(2)}$ as given by Happel & Brenner (1965, p. 305), one obtains

$$\beta_2^{1(2)} = -\frac{4}{9}\left(\frac{a}{R_0}\right)^2 \frac{2+3\alpha}{2(1+\alpha)}\left[U - U_0\left(1-\frac{b^2}{R_0^2}\right)\right] h\left(\frac{b}{R_0}\right) + O\left(\frac{a}{R_0}\right)^4. \tag{44}$$

Substitution of (38) into (44) yields, after some simplifications,

$$\beta_2^{1(2)} = -\frac{4}{9}\left(\frac{a}{R_0}\right)^2 \frac{(\rho_i - \rho_e)\, g a^2}{3\mu_e} h\left(\frac{b}{R_0}\right) + O\left(\frac{a}{R_0}\right)^3, \tag{45}$$

where the function $h(b/R_0)$ is defined as

$$h\left(\frac{b}{R_0}\right) = \frac{3}{2\pi}\sum_{k=-\infty}^{\infty}\int_0^{\infty}\left\{\psi_k(\lambda)\, I_k'(\lambda b) + \pi_k(\lambda)\,[I_k'(\lambda b) + \lambda b\, I_k''(\lambda b)] + \tfrac{1}{2}\omega_k(\lambda)\frac{k I_k(\lambda b)}{\lambda b}\right\}(\lambda R_0)\, d(\lambda R_0), \tag{46}$$

and where $\psi_k(x)$, $\pi_k(\lambda)$ and $\omega_k(\lambda)$ are defined by Happel & Brenner (1965, p. 306) with $H = 1$. I_k is the modified Bessel function of the first kind of order k. I_k' and I_k'' are its first and second derivatives with respect to the entire argument.

Should one want to attempt the second iteration of the velocity field, it is necessary to perform a numerical computation of (46).

For small values of the eccentricity b/R_0, numerical integration of (46) yields (appendix D):

$$h\left(\frac{b}{R_0}\right) = 0{\cdot}598\left(\frac{b}{R_0}\right) + O\left(\frac{b}{R_0}\right)^3.$$

Finally, the only coefficients L_n^m of $O(a/R_0)^2$ are

$$L_2^1 = -\frac{16+19\alpha}{12(1+\alpha)}\left[(\text{ב}) \frac{ab}{R_0^2} + (\text{ב})_s\left(\frac{a}{R_0}\right)^2 h\left(\frac{b}{R_0}\right)\right], \tag{47}$$

$$L_3^0 = \frac{10+11\alpha}{20(1+\alpha)}(\text{ב})\left(\frac{a}{R_0}\right)^2, \tag{48}$$

where

$$(\text{ב}) \equiv \frac{U_0\mu_e}{\sigma}, \quad (\text{ב})_s \equiv \frac{U_{St}\mu_e}{\sigma}, \tag{49}$$

are dimensionless groups and where U_{St} is the Stokes terminal settling velocity.

Finally, the equation of the interface, based on the first iteration, is given by

$$r = a\left[1 + L_2^1 \cos\phi P_2^1(\cos\theta) + L_3^0 P_3(\cos\theta) + O\left(\frac{a}{R_0}\right)^3\right]. \tag{50}$$

Note, that the deviation from sphericity $\xi(\theta, \phi)$ is dependent on the dimensionless groups (ב) and $(\text{ב})_s$ and on the ratios $(a/R_0)^2$, ab/R_0^2. The deviation from sphericity is also dependent on the wall-effect function $h(b/R_0)$.

The magnitude of the deviation function $\xi(\theta, \phi)$ can be determined by substituting the relevant numerical values in equations (46) through (49).

5. Concluding remarks

We have thus computed the flow fields in and around a *spherical* droplet submerged in a Poiseuillian velocity field in a tube. This solution may serve as the first iteration of an iterative procedure for determining more accurate flow fields, taking into account the deviation from sphericity of the interface.

Since the solution presented herein was completed for a spherical droplet, there are no lateral forces acting on the droplet and its velocity is parallel to the axis of the tube. In higher iterations, where deviations from sphericity are taken into account, there will be lateral forces and the droplet may then change its distance b from the axis of the tube.

This work is taken from Haber's (1969) Masters thesis. It was supported by Grant no. 11-1196 from Stiftung Volkswagenwerk, Hanover.

Appendix A

Boundary conditions for the first reflexion (assuming that the droplet is approximately spherical) may be stated as follows:

$$\mathbf{v} \equiv \mathbf{v}^{(0)} + \mathbf{v}^{(1)} = \{U_0[1-(R/R_0)^2 - U]\}\,\mathbf{k}, \quad \text{at} \quad z = \pm\infty, \tag{A 1}$$

and at the interface, i.e. at $r = a$,

$$u_r = 0, \tag{A 2}$$

$$v_r = 0, \tag{A 3}$$

$$u_\theta = v_\theta, \tag{A 4}$$

$$u_\phi = v_\phi, \tag{A 5}$$

$$\tau_{r\theta} = \pi_{r\theta}, \tag{A 6}$$

$$\tau_{r\phi} = \pi_{r\phi}, \tag{A 7}$$

$$\tau_{rr} = \pi_{rr} - \sigma[(1/R_1) + (1/R_2)]\,\mathbf{t}_r, \tag{A 8}$$

where $\tau_{r\theta}$ is the component of the stress tensor acting in the θ direction on an area normal to the r direction.

Now, let a superscript * represent a value on the interface. Then the boundary conditions (A 2)–(A 5) become

$$\mathbf{v}^* = \mathbf{u}^*.$$

Furthermore, boundary condition (A 8) is rewritten using the expression derived by Landau & Lifshitz (1959, p. 239) and (39):

$$\boldsymbol{\tau}^*_{(r)} = \boldsymbol{\pi}^*_{(r)} - \frac{\sigma}{a}\left[2 - 2\xi - \frac{1}{\sin^2\theta}\frac{\partial^2\xi}{\partial\phi^2} - \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\xi}{\partial\theta}\right)\right]\mathbf{t}_r. \tag{A 9}$$

From these vectorial equations we immediately obtain the following relations:

$$\mathbf{v}^*.\mathbf{t}_r = \mathbf{u}^*.\mathbf{t}_r;$$

also,
$$r\nabla.\mathbf{v}^* = r\nabla.\mathbf{u}^*,$$
$$\mathbf{r}.\nabla\times\mathbf{v}^* = \mathbf{r}.\nabla\times\mathbf{u}^*,$$
$$\mathbf{r}.\nabla\times(\mathbf{r}\times\boldsymbol{\pi}^*_{(r)}) = \mathbf{r}.\nabla\times(\mathbf{r}\times\boldsymbol{\tau}^*_{(r)}),$$
$$\mathbf{r}.\nabla\times\boldsymbol{\pi}^*_{(r)} = \mathbf{r}.\nabla\times\boldsymbol{\tau}^*_{(r)}.$$
Now, since
$$\nabla.\mathbf{v} = 0,$$
we obtain
$$-r\nabla.\mathbf{v}^* = \left(r\frac{\partial v_r}{\partial r}\right)^*;$$
also,
$$\mathbf{r}.\nabla\times\mathbf{v}^* = [\mathbf{r}.\nabla\times\mathbf{v}]^*,$$
$$\mathbf{r}.\nabla\times(\mathbf{r}\times\boldsymbol{\tau}^*_{(r)}) = [\mathbf{r}.\nabla\times(\mathbf{r}\times\boldsymbol{\tau}_{(r)})]^*,$$
$$\mathbf{r}.\nabla\times\boldsymbol{\pi}^*_{(r)} = [\mathbf{r}.\nabla\times\boldsymbol{\pi}_{(r)}]^*,$$
and
$$\boldsymbol{\pi}^*_{(r)}.\mathbf{t}_r = \pi^*_{rr}.$$
Similar relations are obtainable for the interior velocity $\mathbf{u}$ and stress vector $\boldsymbol{\tau}_{(r)}$.

From the vectorial relations and the set of boundary conditions (A 2)–(A 9) the following boundary conditions are obtained:
$$u^*_r = v^*_r = 0, \tag{A 10, 11}$$
$$\left[r\frac{\partial u_r}{\partial r}\right]^* = \left[r\frac{\partial v_r}{\partial r}\right]^*, \tag{A 12}$$
$$[\mathbf{r}.\nabla\times\mathbf{u}]^* = [\mathbf{r}.\nabla\times\mathbf{v}]^*, \tag{A 13}$$
$$[\mathbf{r}.\nabla\times(\mathbf{r}\times\boldsymbol{\tau}_{(r)})]^* = [\mathbf{r}.\nabla\times(\mathbf{r}\times\boldsymbol{\pi}_{(r)})]^*, \tag{A 14}$$
$$[\mathbf{r}.\nabla\times\boldsymbol{\tau}_{(r)}]^* = [\mathbf{r}.\nabla\times\boldsymbol{\pi}_{(r)}]^*, \tag{A 15}$$
and
$$\tau^*_{rr} = \pi^*_{rr} - \frac{\sigma}{a}\left[2-2\xi-\frac{1}{\sin^2\theta}\frac{\partial^2\xi}{\partial\phi^2}-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\xi}{\partial\theta}\right)\right]. \tag{A 16}$$

These boundary conditions are to replace the set (A 2) to (A 8). The advantage of rewriting the boundary conditions in this form is that the equations become separated into groups of independent equations which are easily solved. Note that any further vectorial relations must be a linear combination of the seven boundary conditions (A 10) to (A 16).

Appendix B

The stress vector acting on a unit surface of the interface in the r direction, based on the internal velocity, was shown by Lamb (1945) to be
$$\boldsymbol{\tau}_{(r)} = -\frac{\mathbf{r}}{r}p'_i + \mu_i\left(\frac{\partial\mathbf{u}}{\partial r}-\frac{\mathbf{u}}{r}\right)+\frac{\mu_i}{r}\nabla(\mathbf{r}.\mathbf{u}), \tag{B 1}$$
where p'_i is the internal pressure, not including the potential of the body forces, namely
$$p'_i = p_i + g\rho_i r\cos\theta. \tag{B 2}$$

Substitution of (20) and (21) into (B 1) yields, after some simplifications
$$\boldsymbol{\tau}_{(r)} = -(p_0+\rho_i gr\cos\theta)\,\mathbf{t}_r + \frac{\mu_i}{r}\sum_{n=1}^{\infty}\left[(n-1)\nabla\times(\mathbf{r}\chi_n)+2(n-1)\nabla\phi_n - \frac{2n^2+4n+3}{\mu_i(n+1)(2n+3)}\mathbf{r}p_n + \frac{n(n+2)}{\mu_i(n+1)(2n+3)}r^2\nabla p_n\right]. \tag{B 3}$$

A similar procedure can be adopted for calculating the stress vector based on the velocity exterior to the droplet,

$$\boldsymbol{\pi}_{(r)} = -\frac{\mathbf{r}}{r}p_e' + \mu_e\left(\frac{\partial \mathbf{v}}{\partial r} - \frac{\mathbf{v}}{r}\right) + \frac{\mu_e}{r}\nabla(\mathbf{r}\,.\,\mathbf{v}), \tag{B 4}$$

where p_e' is the pressure external to the droplet, not including the potential of body forces, and $\mathbf{v} = \mathbf{v}^{(0)} + \mathbf{v}^{(1)}$.

Substitution of equations (11), (18) and (19) into (B 4) yields

$$\boldsymbol{\pi}_{(r)} = -(p_{-1}+p_\infty)\,\mathbf{t}_r + \mu_e\left(\frac{\partial \mathbf{v}_\infty}{\partial r} - \frac{\mathbf{v}_\infty}{r}\right) + \frac{\mu_e}{r}\nabla(\mathbf{r}\,.\,\mathbf{v}_\infty)$$
$$-\frac{\mu_e}{r}\sum_{n=1}^{\infty}\left[(n+2)\,\nabla\times(\mathbf{r}\chi_{-n-1}) + 2(n+2)\,\nabla\Phi_{-n-1} + \frac{2n^2+1}{\mu_e n(2n-1)}\,\mathbf{r}p_{-n-1}\right.$$
$$\left. - \frac{n^2-1}{\mu_e n(2n-1)}\,r^2\nabla p_{-n-1}\right]. \tag{B 5}$$

Substituting (5*b*) into (B 5) yields finally

$$\boldsymbol{\pi}_{(r)} = -\left[p_{-1} + \left(\rho_0 g - \frac{4U_0\mu_e}{R_0^2}\right) r\cos\theta\right]\mathbf{t}_r - \frac{2\mu_e U_0 \sin\theta}{R_0^2}\,[r(\cos 2\theta\,\mathbf{t}_\theta + \sin 2\theta\,\mathbf{t}_r)$$
$$+ 2b\cos\phi(\cos\theta\,\mathbf{t}_r - \sin\theta\,\mathbf{t}_\theta)] - \frac{2b\mu_e U_0}{R_0^2}(\cos\theta\,\mathbf{t}_\theta - \cos\theta\sin\phi\,\mathbf{t}_\phi)$$
$$-\frac{\mu_e}{r}\sum_{n=1}^{\infty}\left[(n+2)\,\nabla\times(\mathbf{r}\chi_{-n-1}) + 2(n+2)\,\nabla\Phi_{-n-1} + \frac{2n^2+1}{\mu_e n(2n-1)}\,\mathbf{r}p_{-n-1}\right.$$
$$\left. - \frac{(n^2-1)}{\mu_e n(2n-1)}\,r^2\nabla p_{-n-1}\right].$$

Appendix C

It was shown by Hetsroni & Haber (1969) that

$$v_{\infty r} = \mathbf{v}_\infty\,.\,\mathbf{t}_r = \sum_{n=1}^{\infty}\left[\alpha_n\left(\frac{r}{a}\right)^{n+1} + \beta_n\left(\frac{r}{a}\right)^{n-1} + \alpha_{-n-1}\left(\frac{r}{a}\right)^{-n} + \beta_{-n-1}\left(\frac{r}{a}\right)^{-n-2}\right] S_n(\theta,\phi). \tag{C 1}$$

If $\mathbf{v}_\infty$ has no singularities $\alpha_{-n-1} = \beta_{-n-1} = 0$,

then $$\frac{\partial v_{\infty r}}{\partial r} = \sum_{n=1}^{\infty}\left[\frac{\alpha_n}{a}(n+1)\left(\frac{r}{a}\right)^n + \frac{\beta_n}{a}(n-1)\left(\frac{r}{a}\right)^{n-2}\right] S_n(\theta,\phi), \tag{C 2}$$

therefore $$\left[\frac{\partial v_{\infty r}}{\partial r}\right]_{r=0} = a^{-1}\beta_2 S_2(\theta,\phi), \tag{C 3}$$

but $$\mathbf{t}_r\mathbf{t}_r : [\nabla\mathbf{v}_\infty]_{r=0} = \left[\frac{\partial v_{\infty r}}{\partial r}\right]_{r=0}; \tag{C 4}$$

thus $$\mathbf{t}_r\mathbf{t}_r : [\nabla\mathbf{v}_\infty]_0 = a^{-1}\beta_2 S_2(\theta,\phi). \tag{C 5}$$

Appendix D

The function $h(b/R_0)$ was defined in (46) as follows:

$$h\left(\frac{b}{R_0}\right) = \frac{3}{4\pi}\sum_{k=-\infty}^{\infty}\int_0^{\infty}\left\{2\psi_k(\lambda)\,I_k'(\lambda b) + 2\pi_k(\lambda)\,[I_k'(\lambda b) + \lambda b I_k''(\lambda b)] + \omega_k(\lambda)\,\frac{kI_k(\lambda b)}{\lambda b}\right\}(\lambda R_0)\,d(\lambda R_0). \quad \text{(D 1)}$$

In Happel & Brenner (1965, p. 310), the eccentricity function for the torque $g(b/R_0)$ is defined, from which it is easy to derive the following expression

$$g\left(\frac{b}{R_0}\right) = \frac{3}{4\pi}\sum_{k=-\infty}^{\infty}\int_0^{\infty}\left\{\frac{k\omega_k(\lambda)\,I_k(\lambda b)}{\lambda b} - 2\pi_k(\lambda)\,I_k'(\lambda b)\right\}(\lambda R_0)\,d(\lambda R_0), \quad \text{(D 2)}$$

where $\omega_k(\lambda)$, $\pi_k(\lambda)$, $\psi_k(\lambda)$ are defined in Happel & Brenner (1965, p. 306) with $H = 1$.

Thus

$$h\left(\frac{b}{R_0}\right) = g\left(\frac{b}{R_0}\right) + \frac{3}{2\pi}\sum_{k=-\infty}^{\infty}\int_0^{\infty}\{\psi_k(\lambda)\,I_k'(\lambda b) + 2\pi_k(\lambda)\,I_k'(\lambda b) + \pi_k(\lambda)\,\lambda b I_k''(\lambda b)\}(\lambda R_0)\,d(\lambda R_0) = g\left(\frac{b}{R_0}\right) + i\left(\frac{b}{R_0}\right), \quad \text{(D 3)}$$

where

$$i\left(\frac{b}{R_0}\right) = \frac{3}{2\pi}\sum_{k=-\infty}^{\infty}\int_0^{\infty}\{\psi_k(\lambda)\,I_k'(\lambda b) + 2\pi_k(\lambda)\,I'(\lambda b) + \pi_k(\lambda)\,\lambda b I_k''(\lambda b)\}(\lambda R_0)\,d(\lambda R_0). \quad \text{(D 4)}$$

Thus, it suffices to evaluate $i(b/R_0)$, since $g(b/R_0)$ is known and tabulated. In the present work we calculate the function $i(b/R_0)$ only for small distances from the centreline, i.e. $b/R_0 \rightarrow 0$.

Let $b/R_0 = \beta$ and $\lambda R_0 = \delta$ and recall the identity,

$$I_\lambda''(\lambda R_0) = -\frac{I_k'(\lambda R_0)}{\lambda R_0} + \left[1 + \left(\frac{k}{\lambda R_0}\right)^2\right] I_k(\lambda R_0). \quad \text{(D 5)}$$

From (D 4) and (D 5) ,we readily obtain

$$i\left(\frac{b}{R_0}\right) = \frac{3}{2\pi}\sum_{k=-\infty}^{\infty}\int_0^{\infty}\left\{\psi_k(\delta,\delta\beta)\,I_k'(\delta\beta) + \pi_k(\delta,\delta\beta)\left[I_k'(\delta\beta) + \left(\delta\beta + \frac{k^2}{\delta\beta}\right) I_k(\delta\beta)\right]\right\}\delta\,d\delta. \quad \text{(D 6)}$$

Now, if in the above summation only terms with $k = -1, 0, 1$ should be retained, the accuracy would be in the order of β^3, i.e.

$$i\left(\frac{b}{R_0}\right) = \left(\frac{b}{R_0}\right)(\text{Int.}) + O\left(\frac{b}{R_0}\right)^3, \quad \text{(D 7)}$$

where the integral (Int.) is defined as follows:

$$(\text{Int.}) = \frac{3}{2\pi}\int_0^{\infty}\left\{\frac{2\delta}{I_0(\delta)\,I_1(\delta)} - \frac{\delta^2}{2I_0^2(\delta)} - \frac{\delta^2}{2I_1^2(\delta)} + \xi_0(\delta)\,\frac{\delta I_1(\delta)}{I_0(\delta)}\left(1 - \frac{\delta}{2}\frac{I_1(\delta)}{I_0(\delta)}\right) - \xi_1(\delta)\,\frac{\delta^2}{2}\frac{I_2^2(\delta)}{I_1^2(\delta)}\right\}d\delta, \quad \text{(D 8)}$$

and where

$$\xi_0(\delta) = \{I_0^2(\delta) - I_1^2(\delta) - (2/\delta)[I_0(\delta)\,I_1(\delta)]\}^{-1},$$

$$\xi_1(\delta) = \left\{\frac{2I_1^3(\delta)}{\delta^3 I_0(\delta) - \delta^2 I_1(\delta)} + I_1^2(\delta) - I_0^2(\delta) + \frac{2}{\delta^2} I_1^2(\delta)\right\}^{-1}.$$

Numerical integration of equation (D 8) yields the approximation of the deformation function for small values of b/R_0, i.e.

$$h\left(\frac{b}{R_0}\right) = 0{\cdot}598\left(\frac{b}{R_0}\right) + O\left(\frac{b}{R_0}\right)^3.$$

One should also note that for the centreline of the tube, i.e.

$$b/R_0 = 0, \quad i(b/R_0) = g(b/R_0) = 0,$$

therefore the deformation function also vanishes, i.e.

$$h(0) = 0.$$

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